International Journal of Innovation Engineering and Science Research
(ISSN: 2581-4591)

www.ijiesr.com

Formulation of Curious Family of 3-Tuples

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ABSTRACT

This paper deals with the study of formulation of special family of 3-tuples (a,b,c) such that the product of any two elements of the set added with their sum is a perfect square.

Keywords— Diophantine 3-tuples; negative pellian equation; integer solutions.

I. INTRODUCTION

The problem of constructing the sets with property that product of any two of its distinct elements is one less than a square has a very long history and such sets have been studied by Diophantus. A set of m distinct positive integers $\{a_1, a_2, a_3, ...a_m\}$ is said to have the property $D(n), n \in Z - \{0\}$ if $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$ or $1 \le j < i \le m$ and such a set is called a Diophantine m-tuple with property $D(n)$. In this context, one may refer [1-4].

A set of m distinct positive integers $(a_1, a_2, ..., a_m)$ is said to be Dio m-tuple with property $D(n)$ if $a_i a_j + (a_i + a_j) + n$ or $\mathrm{a_i a_j} - (\mathrm{a_i} + \mathrm{a_j}) + \mathrm{n}$ is a perfect square for all $1 \le i < j \le m$ or $1 \le j < i \le m$. In particular, one may refer [5-8] for problem on special dio-3-tuples.

This paper aims at constructing sequences of 3-tuples where the product of any two elements of the set added with their sum is a perfect square.

II. METHOD OF ANALYSIS

Sequence 1:

Let $a = 2k^2 + 6k + 4, c_0 = 8k^2 + 16k + 9$

It is observed that

$$ac_0 + a + c_0 = \left(4k^2 + 10k + 7\right)^2$$

Let c_1 be any integer such that

$$(a+1)c_1 + a = \alpha^2 \tag{1}$$

$$(c_0+1)c_1 + c_0 = \beta^2 \tag{2}$$

Eliminating c_1 between (1) and (2), we have

$$(c_0+1)\alpha^2 - (a+1)\beta^2 = (a - c_0) \tag{3}$$

Introducing the linear transformations

$$\alpha = X + (a+1)T \ , \ \beta = X + (c_0+1)T \tag{4}$$

in (3) and simplifying we get

$$X^2 = (a+1)(c_0+1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 4k^2 + 10k + 7$

In view of (4) and (1), it is seen that

$$c_1 = 18k^2 + 42k + 28$$

Let c_2 be any integer such that

$$(a+1)c_2 + a = \alpha^2 \tag{5}$$

$$(c_1+1)c_2 + c_0 = \beta^2 \tag{6}$$

Eliminating c_2 between (5) and (6), we have

$$(c_1+1)\alpha^2 - (a+1)\beta^2 = (a - c_1) \tag{7}$$

Introducing the linear transformations

$$\alpha = X + (a+1)T \ , \ \beta = X + (c_1+1)T \tag{8}$$

in (7) and simplifying we get

$$X^2 = (a+1)(c_1+1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 6k^2 + 16k + 12$

In view of (8) and (5), it is seen that

$$c_2 = 32k^2 + 80k + 57$$

Let c_3 be any integer such that

$$(a+1)c_3 + a = \alpha^2 \quad (9)$$

$$(c_2+1)c_3 + c_0 = \beta^2 \quad (10)$$

Eliminating c_3 between (9) and (10), we have

$$(c_2+1)\alpha^2 - (a+1)\beta^2 = (a - c_2) \quad (11)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T \ , \ \beta = X + (c_2+1)T \quad (12)$$

in (11) and simplifying we get

$$X^2 = (a+1)(c_2+1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 8k^2 + 22k + 17$

In view of (12) and (9), it is seen that

$$c_3 = 50k^2 + 130k + 96$$

The repetition of the above process leads to the generation of sequence of 3-tuples whose general form is given by (a, c_{s-1}, c_s) where

$$c_{s-1} = (2s^2 + 4s + 2)k^2 + (6s^2 + 8s + 2)k + (5s^2 + 4s) \ , \ s = 1, 2, 3, \ldots$$

A few numerical examples are presented in Table 1 below:

Table 1: Numerical Examples

k	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	(24, 73, 184)	(24, 184, 345)	(24, 345,556)	(24, 556,817)
3	(40, 129, 316)	(40, 316, 585)	(40, 585, 936)	(40, 936, 1369)
4	(60, 201,484)	(60, 484,889)	(60,889,1416)	(60,1416,2065)
5	(84,289,688)	(84,688,1257)	(84,1257,1996)	(84,1996,2905)

Sequence 2:

Let $a = 1, c_0 = 2k^2 - 2k$

It is observed that

$$ac_0 + a + c_0 = (2k-1)^2$$

Let c_1 be any integer such that

$$(a+1)c_1 + a = \alpha^2 \quad (13)$$

$$(c_0+1)c_1 + c_0 = \beta^2 \quad (14)$$

Eliminating c_1 between (13) and (14), we have

$$(c_0+1)\alpha^2-(a+1)\beta^2=(a-c_0) \tag{15}$$

Introducing the linear transformations

$$\alpha = X+(a+1)T \ , \ \beta = X+(c_0+1)T \tag{16}$$

in (15) and simplifying we get

$$X^2=(a+1)(c_0+1)T^2-1$$

which is satisfied by $T=1$, $X=2k-1$

In view of (16) and (13), it is seen that

$$c_1=2k^2+2k$$

Let c_2 be any integer such that

$$(a+1)c_2+a=\alpha^2 \tag{17}$$

$$(c_1+1)c_2+c_0=\beta^2 \tag{18}$$

Eliminating c_2 between (17) and (18), we have

$$(c_1+1)\alpha^2-(a+1)\beta^2=(a-c_1) \tag{19}$$

Introducing the linear transformations

$$\alpha = X+(a+1)T \ , \ \beta = X+(c_1+1)T \tag{20}$$

in (19) and simplifying we get

$$X^2=(a+1)(c_1+1)T^2-1$$

which is satisfied by $T=1$, $X=2k+1$

In view of (20) and (17), it is seen that

$$c_2=2k^2+6k+4$$

Let c_3 be any integer such that

$$(a+1)c_3+a=\alpha^2 \tag{21}$$

$$(c_2+1)c_3+c_0=\beta^2 \tag{22}$$

Eliminating c_3 between (21) and (22), we have

$$(c_2+1)\alpha^2-(a+1)\beta^2=(a-c_2) \tag{23}$$

Introducing the linear transformations

$$\alpha = X+(a+1)T \ , \ \beta = X+(c_2+1)T \tag{24}$$

in (23) and simplifying we get

$$X^2=(a+1)(c_2+1)T^2-1$$

which is satisfied by $T=1$, $X=2k+3$

In view of (24) and (21), it is seen that

$$c_3 = 2k^2 + 10k + 12$$

The repetition of the above process leads to the generation of sequence of 3-tuples whose general form is given by (a, c_{s-1}, c_s) where

$$c_{s-1} = 2k^2 + (4s-6)k + (2s^2 - 6s + 4)\ ,\ s = 1,2,3,...$$

A few numerical examples are presented in Table 2 below:

Table 2: Numerical Examples

k	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	(1, 4, 12)	(1, 12, 24)	(1, 24, 40)	(1, 40, 60)
3	(1, 12, 24)	(1, 24, 40)	(1, 40, 60)	(1, 60, 84)
4	(1, 24,40)	(1, 40, 60)	(1,60, 84)	(1, 84, 112)
5	(1, 40, 60)	(1, 60, 84)	(1, 84, 112)	(1, 112, 144)

Sequence 3:

Let $a = 1, c_0 = 2k^2 + 2k$

It is observed that

$$ac_0 + a + c_0 = (2k+1)^2$$

Let c_1 be any integer such that

$$(a+1)c_1 + a = \alpha^2 \qquad (25)$$

$$(c_0+1)c_1 + c_0 = \beta^2 \qquad (26)$$

Eliminating c_1 between (25) and (26), we have

$$(c_0+1)\alpha^2 - (a+1)\beta^2 = (a - c_0) \qquad (27)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T\ ,\ \beta = X + (c_0+1)T \qquad (28)$$

in (27) and simplifying we get

$$X^2 = (a+1)(c_0+1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 2k+1$

In view of (28) and (25), it is seen that

$$c_1 = 2k^2 + 6k + 4$$

Let c_2 be any integer such that

$$(a+1)c_2 + a = \alpha^2 \qquad (29)$$

$$(c_1+1)c_2 + c_0 = \beta^2 \qquad (30)$$

Eliminating c_2 between (29) and (30), we have

$$(c_1+1)\alpha^2-(a+1)\beta^2=(a-c_1) \quad (31)$$

Introducing the linear transformations

$$\alpha = X+(a+1)T \ , \ \beta = X+(c_1+1)T \quad (32)$$

in (31) and simplifying we get

$$X^2=(a+1)(c_1+1)T^2-1$$

which is satisfied by $T=1$, $X=2k+3$

In view of (32) and (29), it is seen that

$$c_2=2k^2+10k+12$$

Let c_3 be any integer such that

$$(a+1)c_3+a=\alpha^2 \quad (33)$$

$$(c_2+1)c_3+c_0=\beta^2 \quad (34)$$

Eliminating c_3 between (33) and (34), we have

$$(c_2+1)\alpha^2-(a+1)\beta^2=(a-c_2) \quad (35)$$

Introducing the linear transformations

$$\alpha = X+(a+1)T \ , \ \beta = X+(c_2+1)T \quad (36)$$

in (35) and simplifying we get

$$X^2=(a+1)(c_2+1)T^2-1$$

which is satisfied by $T=1$, $X=2k+5$

In view of (36) and (33), it is seen that

$$c_3=2k^2+14k+24$$

The repetition of the above process leads to the generation of sequence of 3-tuples whose general form is given by (a,c_{s-1},c_s) where

$$c_{s-1}=2k^2+(4s-2)k+(2s^2-2s)\ ,\ s=1,2,3,...$$

A few numerical examples are presented in Table 3 below:

Table 3: Numerical Examples

k	(a,c_0,c_1)	(a,c_1,c_2)	(a,c_2,c_3)	(a,c_3,c_4)
2	(1,12,24)	(1, 24, 40)	(1, 40, 60)	(1, 60, 84)
3	(1,24,40)	(1, 40, 60)	(1, 60, 84)	(1, 84, 112)
4	(1,40,60)	(1, 60, 84)	(1, 84, 112)	(1, 112, 144)
5	(1,60,84)	(1, 84, 112)	(1, 112, 144)	(1, 144, 180)

Sequence 4:

Let $a=4, c_0=5k^2+4k$

It is observed that

$$ac_0 + a + c_0 = (5k+2)^2$$

Let c_1 be any integer such that

$$(a+1)c_1 + a = \alpha^2 \qquad (37)$$

$$(c_0+1)c_1 + c_0 = \beta^2 \qquad (38)$$

Eliminating c_1 between (37) and (38), we have

$$(c_0+1)\alpha^2 - (a+1)\beta^2 = (a - c_0) \qquad (39)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T \ , \ \beta = X + (c_0+1)T \qquad (40)$$

in (39) and simplifying we get

$$X^2 = (a+1)(c_0+1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 5k+2$

In view of (40) and (37), it is seen that

$$c_1 = 5k^2 + 14k + 9$$

Let c_2 be any integer such that

$$(a+1)c_2 + a = \alpha^2 \qquad (41)$$

$$(c_1+1)c_2 + c_0 = \beta^2 \qquad (42)$$

Eliminating c_2 between (41) and (42), we have

$$(c_1+1)\alpha^2 - (a+1)\beta^2 = (a - c_1) \qquad (43)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T \ , \ \beta = X + (c_1+1)T \qquad (44)$$

in (43) and simplifying we get

$$X^2 = (a+1)(c_1+1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 5k+7$

In view of (44) and (41), it is seen that

$$c_2 = 5k^2 + 24k + 28$$

Let c_3 be any integer such that

$$(a+1)c_3 + a = \alpha^2 \qquad (45)$$

$$(c_2+1)c_3 + c_0 = \beta^2 \qquad (46)$$

Eliminating c_3 between (45) and (46), we have

$$(c_2+1)\alpha^2-(a+1)\beta^2=(a-c_2) \quad (47)$$

Introducing the linear transformations

$$\alpha = X+(a+1)T \ , \ \beta = X+(c_2+1)T \quad (48)$$

in (47) and simplifying we get

$$X^2=(a+1)(c_2+1)T^2-1$$

which is satisfied by $T=1$, $X=5k+12$

In view of (48) and (45), it is seen that

$$c_3=5k^2+34k+57$$

The repetition of the above process leads to the generation of sequence of 3-tuples whose general form is given by (a,c_{s-1},c_s) where

$$c_{s-1}=5k^2+(10s-6)k+(5s^2-6s+1) \ , \ s=1,2,3,...$$

A few numerical examples are presented in Table 4 below:

Table 4: Numerical Examples

k	(a,c_0,c_1)	(a,c_1,c_2)	(a,c_2,c_3)	(a,c_3,c_4)
2	(4,28,57)	(4, 57, 96)	(4, 96, 145)	(4, 145, 204)
3	(4,57,96)	(4, 96, 145)	(4, 145, 204)	(4, 204, 273)
4	(4,96,145)	(4, 145, 204)	(4, 204, 273)	(4, 273, 352)
5	(4,145,204)	(4, 204, 273)	(4, 273, 352)	(4, 352, 441)

Sequence 5:

Let $a=4, c_0=5k^2-4k$

It is observed that

$$ac_0+a+c_0=(5k-2)^2$$

Let c_1 be any integer such that

$$(a+1)c_1+a=\alpha^2 \quad (49)$$

$$(c_0+1)c_1+c_0=\beta^2 \quad (50)$$

Eliminating c_1 between (49) and (50), we have

$$(c_0+1)\alpha^2-(a+1)\beta^2=(a-c_0) \quad (51)$$

Introducing the linear transformations

$$\alpha = X+(a+1)T \ , \ \beta = X+(c_0+1)T \quad (52)$$

in (51) and simplifying we get

$$X^2=(a+1)(c_0+1)T^2-1$$

which is satisfied by $T=1$, $X=5k-2$

In view of (52) and (49), it is seen that

$$c_1=5k^2+6k+1$$

Let c_2 be any integer such that

$$(a+1)c_2+a=\alpha^2 \quad (53)$$

$$(c_1+1)c_2+c_0=\beta^2 \quad (54)$$

Eliminating c_2 between (53) and (54), we have

$$(c_1+1)\alpha^2-(a+1)\beta^2=(a-c_1) \quad (55)$$

Introducing the linear transformations

$$\alpha=X+(a+1)T \ , \ \beta=X+(c_1+1)T \quad (56)$$

in (55) and simplifying we get

$$X^2=(a+1)(c_1+1)T^2-1$$

which is satisfied by $T=1$, $X=5k+3$

In view of (56) and (53), it is seen that

$$c_2=5k^2+16k+12$$

Let c_3 be any integer such that

$$(a+1)c_3+a=\alpha^2 \quad (57)$$

$$(c_2+1)c_3+c_0=\beta^2 \quad (58)$$

Eliminating c_3 between (57) and (58), we have

$$(c_2+1)\alpha^2-(a+1)\beta^2=(a-c_2) \quad (59)$$

Introducing the linear transformations

$$\alpha=X+(a+1)T \ , \ \beta=X+(c_2+1)T \quad (60)$$

in (59) and simplifying we get

$$X^2=(a+1)(c_2+1)T^2-1$$

which is satisfied by $T=1$, $X=5k+8$

In view of (60) and (57), it is seen that

$$c_3=5k^2+26k+33$$

The repetition of the above process leads to the generation of sequence of 3-tuples whose general form is given by (a,c_{s-1},c_s) where

$$c_{s-1}=5k^2+(10s-14)k+(5s^2-14s+9) \ , \ s=1,2,3,\ldots$$

A few numerical examples are presented in Table 5 below:

Table 5: Numerical Examples

k	(a,c_0,c_1)	(a,c_1,c_2)	(a,c_2,c_3)	(a,c_3,c_4)
2	(4,12,33)	(4, 33, 64)	(4, 64, 105)	(4, 105, 156)
3	(4,33,64)	(4, 64, 105)	(4, 105, 156)	(4, 156, 217)
4	(4,64,105)	(4, 105, 156)	(4, 156, 217)	(4, 217, 288)
5	(4,105,156)	(4, 156, 217)	(4, 217, 288)	(4, 288, 369)

Sequence 6:

Let $a=12, c_0=13k^2-10k+1$

It is observed that

$ac_0+a+c_0=(13k-5)^2$

Let c_1 be any integer such that

$$(a+1)c_1+a=\alpha^2 \quad (61)$$

$$(c_0+1)c_1+c_0=\beta^2 \quad (62)$$

Eliminating c_1 between (61) and (62), we have

$$(c_0+1)\alpha^2-(a+1)\beta^2=(a-c_0) \quad (63)$$

Introducing the linear transformations

$$\alpha=X+(a+1)T \;,\; \beta=X+(c_0+1)T \quad (64)$$

in (63) and simplifying we get

$$X^2=(a+1)(c_0+1)T^2-1$$

which is satisfied by $T=1$, $X=13k-5$

In view of (64) and (61), it is seen that

$$c_1=13k^2+16k+4$$

Let c_2 be any integer such that

$$(a+1)c_2+a=\alpha^2 \quad (65)$$

$$(c_1+1)c_2+c_0=\beta^2 \quad (66)$$

Eliminating c_2 between (65) and (66), we have

$$(c_1+1)\alpha^2-(a+1)\beta^2=(a-c_1) \quad (67)$$

Introducing the linear transformations

$$\alpha=X+(a+1)T \;,\; \beta=X+(c_1+1)T \quad (68)$$

in (67) and simplifying we get

$$X^2=(a+1)(c_1+1)T^2-1$$

which is satisfied by $T=1$, $X=13k+8$

In view of (68) and (65), it is seen that

$$c_2 = 13k^2 + 42k + 33$$

Let c_3 be any integer such that

$$(a+1)c_3 + a = \alpha^2 \quad (69)$$

$$(c_2+1)c_3 + c_0 = \beta^2 \quad (70)$$

Eliminating c_3 between (69) and (70), we have

$$(c_2+1)\alpha^2 - (a+1)\beta^2 = (a - c_2) \quad (71)$$

Introducing the linear transformations

$$\alpha = X + (a+1)T \ , \ \beta = X + (c_2+1)T \quad (72)$$

in (71) and simplifying we get

$$X^2 = (a+1)(c_2+1)T^2 - 1$$

which is satisfied by $T = 1$, $X = 13k + 21$

In view of (72) and (69), it is seen that

$$c_3 = 13k^2 + 68k + 88$$

The repetition of the above process leads to the generation of sequence of 3-tuples whose general form is given by (a, c_{s-1}, c_s) where

$$c_{s-1} = 13k^2 + (26s - 36)k + (13s^2 - 36s + 24) \ , \ s = 1,2,3,...$$

A few numerical examples are presented in Table 6 below:

Table 6: Numerical Examples

k	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	(a, c_3, c_4)
2	(12,33,88)	(12, 88, 169)	(12, 169, 276)	(12, 276, 409)
3	(12,88,169)	(12, 169,276)	(12, 276, 409)	(12, 409, 568)
4	(12,169,276)	(12, 276,409)	(12, 409, 568)	(12, 568, 753)
5	(12,276,409)	(12,409,568)	(12, 568,753)	(12, 753, 964)

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