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The scaling invariant spaces for fractional Navier-Stokes equations

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ABSTRACT

In this paper, we consider the scaling invariant spaces for fractional Navier-Stokes in the Lebesgue spaces $L^p(R^n)$ and homogeneous Besov spaces $\dot{B}^s_{p,q}(R^n)$ respectively.

Keywords—*scaling invariant spaces; fractional Navier-Stokes; parameters; Besov spaces*

I. Introduction

In this note, we study the scaling invariant spaces of the fractional Navier-Stokes equations (also called generalized Navier-Stokes equations) on the half-sapce $R_+^{1+n}=(0,\infty)\times R^n, n\geq 2$:

$$\begin{cases} u_t+(-\Delta)^{\beta}u+(u\cdot\nabla)u-\nabla\pi=0, R_+^{1+n}, \\ \nabla\cdot u=0, R_+^{1+n}, \\ u(x,0)=u_0, R_+^{1+n}, \end{cases} \quad (1)$$

where $\beta\in(\frac{1}{2},1)$. The fractional Navier-Stokes equations (1) has been studied by many authors. Lions [1] obtain the global existence of the classical solutions when $\beta\geq\frac{5}{4}$ in the 3D case. Wu [2] got the n dimension result for $\beta\geq\frac{1}{2}+\frac{n}{4}$, in [3] considered the existence of solution in $\dot{B}_{p,q}^{1+\frac{n}{p}-2\beta}(R^n)$. There are many other results in [4-8] and the reference there.

In this paper, we mainly study the road of finding the scaling invariant spaces for fractional Navier-Stokes equations in Lebesgue space $L^p(R^n)$ and the homogeneous Besov space $\dot{B}^s_{p,q}(R^n)$, where the space $L^p(R^n)$ is the set of function f satisfying

$$\|f\|_{L^p(R^n)}=\left(\int_{R^n}|f(x)|^p\,dx\right)^{\frac{1}{p}}<\infty, 0<p<\infty,$$

and the homogeneous Besov space is the subset of the dual of the Schwartz space $S'(R^n)$, with the boundedness of the semi norm

$$\|f\|_{\dot{B}^s_{p,q}(R^n)}=\left(\sum_{j\in Z}2^{qjs}\left\|\dot{\Delta}_j f\right\|_p^q\right)^{\frac{1}{q}}<\infty.$$

II. RESULTS AND PROOFS

Before we give our main theorem, we firstly give a lemma which will be used later.

Lemma 2.1 (The scaling invariant spaces) The scaling invariant spaces satisfy

$$u_\lambda(t,x)=\lambda^{2\beta-1}u(\lambda^{2\beta}t,\lambda x)\ ,\ \pi_\lambda(t,x)=\lambda^{4\beta-2}u(\lambda^{2\beta}t,\lambda x),(u_0)_\lambda(x)=\lambda^{2\beta-1}u_0(\lambda x).$$

Proof: We firstly proof the scaling transforms of the functions $u(t,x)$, $\pi(t,x)$, $u_0(x)$ are $u_\lambda(t,x)=\lambda^a u(\lambda^b t,\lambda^c x)$, $\pi_\lambda(t,x)=\lambda^d\pi(\lambda^e t,\lambda^f x),(u_0)_\lambda(x)=\lambda^g u_0(\lambda^h x)$,

where a,b,c,d,e,f,g,h are non-negative integers to be determined later. If (u,π,u_0) are the solution of the system (1), then we take $(u_\lambda,\pi_\lambda,(u_0)_\lambda)$ into the system (1) and find the relationships between a,b,c,d,e,f,g,h such that $(u_\lambda,\pi_\lambda,(u_0)_\lambda)$ are also the solution of the system (1).

We calculate that

$$(u_\lambda)_t=\lambda^{a+b}u_t(\lambda^b t,\lambda^c x)\ ,$$
$$(-\Delta)^\beta u_\lambda=\lambda^a\cdot\lambda^{2\beta c}(-\Delta)^\beta u\,,$$
$$\nabla\cdot u_\lambda=\lambda^{a+b}\nabla\cdot u(\lambda^b t,\lambda^c x),$$
$$\nabla\pi_\lambda=\lambda^{d+f}\nabla\pi(\lambda^e t,\lambda^f x).$$

Putting all the equations above into the first equation of the system (1), we have

$$\lambda^{a+b}u_t(\lambda^b t,\lambda^c x)+\lambda^{a+2\beta c}(-\Delta)^\beta u+\lambda^a\cdot\lambda^{a+c}u(\lambda^b t,\lambda^c x)\cdot\nabla u(\lambda^b t,\lambda^c x)-\lambda^{d+f}\nabla\pi(\lambda^e t,\lambda^f x)=0,$$

For the aim that $(u_\lambda,\pi_\lambda,(u_0)_\lambda)$ are also the solution of the first equation of the system (1), we need that

$$a+b=a+2\beta c=2a+c=d+f\ .$$

We note that the above equations have 3 equations with 6 unknown variables, there are infinity solutions with 3 free variables. And through computing, we have

$$a+b=a+2\beta c\ \Rightarrow b=2\beta c\,,$$
$$a+b=2a+c\ \Rightarrow b=a+c\,,$$

here we can choose $c=1$, thus $b=2\beta$ and $a=2\beta-1$. After that we take $f=1$, due to $a+b=d+f$, that is $2\beta-1+2\beta=d+1$, we have $d=4\beta-2$. The variable e can be arbitrary.

Since the term π can be expressed by u , we know that the important work of the determination of the scaling invariant spaces is to choose the parameters in $u_\lambda(t,x)$, that is the determination of the parameters a,b,c,d,e,f,g,h. The method is by the fact that if the function $u(t,x)$ satisfies the system (1), so does $u_\lambda(t,x)$, thus we determine the parameters in the scaling invariant spaces.

Next, we obtain the scaling invariant spaces X for the system (1), that is we find the spaces X , such that $\|u\|_X=\|u_\lambda\|_X$, where $u_\lambda(x)=\lambda^{2\beta-1}u(\lambda x)$. We consider the cases that X is the Lebesgue space $L^p(R^n)$ and the homogeneous Besov space $\dot{B}^s_{p,q}(R^n)$ respectively. The first result is that X is of the Lebesgue space $L^p(R^n)$.

Theorem 2.1 Fractional Navier Stokes equations (1) are scaling invariant on $L^p(R^n)$, if and only if $p=\dfrac{n}{2\beta-1}$.

Proof: It is sufficient to show that $\|u\|_{L^p(R^n)}=\|u_\lambda\|_{L^p(R^n)}$. Due to $u_\lambda(x)=\lambda^{2\beta-1}u(\lambda x)$, we have

$$\|u_\lambda\|_{L^p(R^n)}=\left(\int_{R^n}|\lambda^{2\beta-1}u(\lambda x)|^p\,dx\right)^{\frac{1}{p}}.$$

Set $\lambda x = x'$, thus $x = \frac{x'}{\lambda}$, $dx = \frac{1}{\lambda^n} dx'$, we get

$$\|u_\lambda\|_{L^p(R^n)} = \left(\int_{R^n} |\lambda^{2\beta-1} u(x')|^p \frac{1}{\lambda^n} dx' \right)^{\frac{1}{p}}$$
$$= \lambda^{2\beta-1-\frac{n}{p}} \left(\int_{R^n} |u(x')|^p \, dx' \right)^{\frac{1}{p}} = \lambda^{2\beta-1-\frac{n}{p}} \|u\|_{L^p(R^n)},$$

therefore, to make sure $\|u\|_{L^p(R^n)} = \|u_\lambda\|_{L^p(R^n)}$ to be true, we need $2\beta - 1 - \frac{n}{p} = 0$, that is $p = \frac{n}{2\beta - 1}$. So we have the proof done.

Then, we show the result that X is of the homogeneous Besov space $\dot{B}^s_{p,q}(R^n)$.

Theorem 2.2 Fractional Navier Stokes equations (1) are scaling invariant on $\dot{B}_{p,q}^{-(2\beta-1)+\frac{n}{p}}(R^n)$.

Proof: by the definition

$$\Delta_j u_\lambda(x) = \int_{R^n} \varphi_j(y) \lambda^{2\beta-1} u(\lambda(x-y)) dy$$
$$= \int_{R^n} \varphi_j(y) \lambda^{2\beta-1} u(\lambda x - \lambda y) dy,$$

taking the change of variable $\lambda y = y'$, that is $dy = \frac{1}{\lambda^n} dy'$, we have

$$\Delta_j u_\lambda(x) = \int_{R^n} \varphi_j(\frac{y'}{\lambda}) \lambda^{2\beta-1} u(\lambda x - y') dy'$$
$$= \lambda^{2\beta-1} \int_{R^n} \varphi_j(\frac{y'}{\lambda}) u(\lambda x - y') dy',$$

where

$$\varphi_j(\frac{y'}{\lambda}) = \int_{R^n} \phi(2^{-j}\xi) e^{2\pi i \frac{y'}{\lambda}\cdot\xi} d\xi = \int_{R^n} \phi(2^{-j}\lambda\xi') e^{2\pi i y'\cdot\xi'} \lambda^n d\xi'$$
$$= \lambda^n \int_{R^n} \phi(2^{-j}\lambda\xi') e^{2\pi i y'\cdot\xi'} d\xi',$$

where $\frac{\xi}{\lambda} = \xi'$, so we have

$$\varphi_j(\frac{y'}{\lambda}) = \lambda^n \int_{R^n} \phi(2^{-j}\lambda\xi') e^{2\pi i y'\cdot\xi'} d\xi',$$

Taking $2^{-j}\lambda = 2^{-j'}$, we get

$$j' = -\log_2 2^{-j\lambda} = -(\log_2 2^{-j} + \log_2 \lambda) = j - \log_2 \lambda.$$

Therefore, we obtain

$$\varphi_j(\frac{y'}{\lambda}) = \lambda^n \int_{R^n} \phi(2^{-j}\xi) e^{2\pi i y'\cdot\xi} d\xi = \lambda^n \varphi_{j'}(y'),$$

which implies $\Delta_j u_\lambda(x) = \lambda^{2\beta-1-n} \int_{R^n} \lambda^n \varphi_j(y') u(\lambda x - y') dy'$. As a result,

$$\|\Delta_j u_\lambda(x)\|_{L^p(R^n)} = \left(\int_{R^n} |\lambda^{2\beta-1} \int_{R^n} \varphi_{j'}(y') u(\lambda x - y') dy'|^p \, dx \right)^{\frac{1}{p}}$$
$$= \left(\int_{R^n} |\lambda^{2\beta-1} \int_{R^n} \varphi_{j'}(y) u(\lambda x - y) dy|^p \, dx \right)^{\frac{1}{p}}.$$

Taking a change of variable $\lambda x = x'$, that is hence $x = \frac{x'}{\lambda}, dx = \frac{1}{\lambda^n} dx'$, hence

$$\left\|\Delta_j u_\lambda(x)\right\|_{L^p(R^n)} = \left(\int_{R^n} |\lambda^{2\beta-1}\int_{R^n}\varphi_{j'}(y)u(x'-y)dy|^p \frac{1}{\lambda^n}dx'\right)^{\frac{1}{p}}$$

$$= \lambda^{2\beta-1-\frac{n}{p}}\left(\int_{R^n}|\int_{R^n}\varphi_{j'}(y)u(x'-y)dy|^p\, dx'\right)^{\frac{1}{p}} = \lambda^{2\beta-1-\frac{n}{p}}\left\|\Delta_{j'}u\right\|_{L^p(R^n)}.$$

By the definition of the norm of Besov spaces,

$$\|f\|_{\dot{B}^s_{p,q}(R^n)} = \left(\sum_{j=-\infty}^{\infty}(2^{sj}\left\|\Delta_j f\right\|_{L^p(R^n)})^q\right)^{\frac{1}{q}},$$

we have $\|f_\lambda\|_{\dot{B}^s_{p,q}(R^n)} = \left(\sum_{j=-\infty}^{\infty}(2^{sj}\left\|\Delta_j f_\lambda\right\|_{L^p(R^n)})^q\right)^{\frac{1}{q}}$, by the conclusion that

$$\left\|\Delta_j f_\lambda\right\|_{L^p(R^n)} = \lambda^{2\beta-1-\frac{n}{p}}\left\|\Delta_{j'} f\right\|_{L^p(R^n)},$$

where $j' = j - \log_2\lambda$. Thus, the norm of u_λ in $\dot{B}^s_{p,q}(R^n)$ is

$$\left\|u_\lambda\right\|_{\dot{B}^s_{p,q}(R^n)} = \left(\sum_{j=-\infty}^{\infty}(2^{sj}\lambda^{2\beta-1-\frac{n}{p}}\left\|\Delta_{j'}u\right\|_{L^p(R^n)})^q\right)^{\frac{1}{q}}$$

$$= \left(\sum_{j=-\infty}^{\infty}(2^{sj'}\lambda^s\lambda^{2\beta-1-\frac{n}{p}}\left\|\Delta_{j'}u\right\|_{L^p(R^n)})^q\right)^{\frac{1}{q}}$$

$$= \lambda^{s+2\beta-1-\frac{n}{p}}\left(\sum_{j'=-\infty}^{\infty}(2^{sj'}\left\|\Delta_{j'}u\right\|_{L^p(R^n)})^q\right)^{\frac{1}{q}}$$

$$= \lambda^{s+2\beta-1-\frac{n}{p}}\left\|u\right\|_{\dot{B}^s_{p,q}(R^n)},$$

where we used $j = j' + \log_2\lambda$ and $sj = sj' + s\log_2\lambda$, therefore

$$2^{sj} = 2^{sj'+s\log_2\lambda} = 2^{sj'}2^{s\log_2\lambda} = 2^{sj'}\lambda^s.$$

To make sure $\left\|u_\lambda\right\|_{\dot{B}^s_{p,q}(R^n)} = \left\|u\right\|_{\dot{B}^s_{p,q}(R^n)}$, we need

$$s + 2\beta - 1 - \frac{n}{p} = 0 \Rightarrow s = -(2\beta-1) + \frac{n}{p}.$$

It follows that the homogeneous Besov space should be chosen as $\dot{B}^s_{p,q}(R^n) = \dot{B}^{-(2\beta-1)+\frac{n}{p}}_{p,q}(R^n)$. Consequently we have the proof done.

By the embedding theorem of the homogeneous Besov spaces, we know that when p, q are infinity, the space is the biggest one $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(R^n)$. And if $\beta = 1$, the system (1) becomes Navier-Stokes equations, the corresponding scaling invariant space is $\dot{B}^{-1}_{\infty,\infty}(R^n)$. If $\beta = 0$, the system (1) correspond to Euler equations, then the corresponding scaling invariant space is $\dot{B}^1_{\infty,\infty}(R^n)$.

III. CONCLUSIONS

We consider the value of the index parameters p in Lebesgue spaces $L^p(R^n)$, and s, p, q in homogeneous Besov spaces $\dot{B}^s_{p,q}(R^n)$ for fractional Navier-Stokes equations to be scaling invariant in these spaces. We conclude that, the parameter p in Lebesgue spaces must be $p = \frac{n}{2\beta - 1}$, and the homogeneous Besov space must be $\dot{B}^{-(2\beta-1)+\frac{n}{p}}_{p,q}(R^n)$. Due to the embedding theorem in homogeneous spaces, we know for fractional Navier-Stokes equations the biggest scaling invariant homogeneous Besov space is $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(R^n)$. And as the parameter special cases, we know the biggest s homogeneous Besov space for Navier-Stokes equations is $\dot{B}^{-1}_{\infty,\infty}(R^n)$, the one for Euler equations is $\dot{B}^{1}_{\infty,\infty}(R^n)$.

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