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On Carbon Emission Credits Options Pricing

Richard OnyinoSimwa

Division of Actuarial Science& Financial Mathematics,

School of Mathematics, University of Nairobi

ABSTRACT

The effect of adverse climate change is of major concern worldwide and several approaches are being developed to mitigate against anticipated economic and social disaster. Carbon emissions has been identified as a major contributor to the adverse climate change and following the Kyoto protocol , European countries have , through a caucus, effected a market to reward or fine members depending on their compliance position. The commodity for the market is the carbon emission credits. Stochastic models for pricing of options on these credits are considered in this paper. In particular, we determine the price basing on the Normal Inverse Gaussian and the Brownian Motion models. Maximum Likelihood Estimation is applied to determine model parameter estimates in each case. It is shown that the Normal Inverse Gaussian model has a better fit to the data but gives higher prices for a given strike price , compared to the Brownian Motion model.

Key Words: Carbon Emission Credit, Brownian Motion, Kyoto Protocol Compliance, Normal Inverse Gaussian Distribution, Fourier Transform, Risk-Neutral Option Pricing.

I. INTRODUCTION

Greenhouse gas (GHG) emissions come from the burning of fossil fuels for energy (e.g. for electricity and transport). When oil, gas or coal burns, carbon contained within it combines with oxygen in the air to create carbon dioxide. Globally, almost 80% of GHG emissions come from human sources. Global GHG emissions grew by approximately 42% between 1990 and 2011, with the bulk growth occurring in emerging markets and developing countries.

The release of GHGs and their increasing concentration in the atmosphere are already having an impact on the environment, human health and the economy. These impacts are expected to become more severe, unless concerted efforts to reduce emissions are undertaken. Environmental impacts include the following. Annual temperatures are expected to rise, increased coastal flooding due to increased temperatures, heat waves that could result to forest fires and Shrinking water supplies. Human health impacts include high temperatures that may increase the risk of deaths from dehydration and heat stroke, an increase in water-, food-, vector- and rodent-borne diseases, cancer diseases that develop due to air pollution. Economic impacts include agriculture, forestry, tourism and recreation being affected by changing weather patterns and damage to infrastructure caused by extreme weather events.

In order to address climate change globally, the Kyoto Protocol was introduced. The Kyoto Protocol is an international agreement linked to the United Nations Framework Convention on Climate Change, which commits its parties by setting international binding emission reduction targets.

The Kyoto Protocol was adopted in Kyoto, Japan, on 11 December 1997 and entered into force on 16th February, 2005. Participating countries that have ratified the Kyoto Protocol have committed to cut GHG emissions. The Kyoto Protocol sets binding emission reduction targets for participants. The goal of Kyoto was to see participants collectively reducing emissions of greenhouse gases by 5.2% below the emission levels of 1990 by 2012.

Binding emissions reduction commitment for participants meant that the space to pollute was limited, and what is scarce and essential commands a price. Greenhouse gas emissions- most prevalently carbon dioxide- became a new commodity. Kyoto Protocol now began to internalize what was now recognized as an unpriced externality. Since carbon dioxide is the principle greenhouse gas, people speak simply of trading in carbon.

This leads us to the second, the flexible mechanisms of the Kyoto Protocol, based on the trade of emissions permits. Kyoto Protocol countries bound to targets have to meet them largely through domestic action- that is, to reduce their emissions onshore. But they can meet part of their targets through the "market-based mechanisms" that ideally encourage GHG abatement to start where it is most cost-effective- for example, in the developing world. Quite simply, it does not matter where emissions are reduced, as long as they are removed from the planet's atmosphere. The Kyoto mechanisms are, InternationalEmissions Trading (IET), Clean Development Mechanism (CDM) and Joint Implementation (JI).

Parties with commitments under the Kyoto Protocol that have accepted targets for limiting or reducing emissions are called Annex I parties**.** These countries are set a legally binding cut for GHG emission to 5.2% below their 1990 level. This reduction is to be attained in sum over all Annex I members over a five year compliance period which is from 2008 to 2012. The concrete implementation is as follows: Each Annex I member is assigned a certain Carbon dioxide gas amount, which equals to (5 years)*(country's emission in 1990)*(1-0.0052). This credit is measured in the so- called Assigned Amount Units (AAUs), corresponding to one ton of carbon dioxide. Each member faces penalties if its entire emission within the compliance period 2008-2012 exceeds member's total number of AAUs.

The IET mechanism allows annex I members that have emission units to spare to sell the excess to annex I members that are over their targets. Therefore, the carbon allowances traded in IET mechanism are called AAUs.

The CDMmechanism allows an Annex I member to implement an emission-reduction project in developing countries. Such projects can earn saleable certified emission reduction (CER) credits, each equivalent to one tonne of Carbon dioxide gas, which can be counted towards meeting Kyoto targets.

II. LITERATURE REVIEW

Carmona and Hinz(2011) examined the spot EUA returns that exhibit a volatility clustering feature and the carbon market system that is impacted by the announcements of CO_2 emissions policies. They proposed a regime-switching jump diffusion model (RSJM) with a hidden Markov chain to capture not only a volatility clustering feature, but also the dynamics of the spot EUA returns that are influenced by change in the CO_2 emissions policies, and thereby altering jump arrivals. They concluded that RSJM is the best model to fit the price behavior of the carbon markets and to price its related derivatives.

Chevallier, J. and Sevi B. (2014) argue that jumps need to be explicitly taken into account when modeling spot and future carbon price series. He says that there is evidence that suggests that carbon futures are a pure jump process without a continuous component and a relatively high activity index. The occurrence of jumps on the carbon market may be related to information disclosure about allocation, or changes in the perimeter of the scheme. He concludes that derivatives models such as the models by Cont and Tankov (2004), and the CGMY model (Carr et al. (2002)) would be plausible candidates, since these models accommodate a pure-jump stochastic process with activity indices above unity, as found in the carbon futures data.

Seifert et al(2007) explain that European call and put options are actively traded on EUA future contracts. Since 2006, trades of options maturing in December of each year (prior to 2012) have produced a term structure of option prices. It is noted that, whether or not traders are using Black-Scholes to price options on EUAs and future contracts, it is important to have option price formulas based on underlying martingale with binary terminal value.

Bolviken and Benth(2000) argued that traders of carbon emissions need to have a valid carbon dioxide spot price model so that it can be possible to value potential derivatives and so that carbon emitting companies can be able to better assess their production costs and support emissions-related investment decisions. They therefore presented a tractable stochastic equilibrium model reflecting stylized features of the emissions trading scheme and analyzed the resulting carbon spot price dynamics. Their main findings were that carbon prices do not have to follow any seasonal patterns, discounted prices and should possess the martingale property, and an adequate carbon price process should exhibit a time- and price- dependant volatility structure.

Carmona and Hinz (2011) argue that martingales finishing at two-valued random variables can be considered as basic building blocks which form the risk-neutral futures price dynamics. They therefore suggest a model for two-valued martingales, flexible in terms of time- and space changing volatility and capable to match the observed historical or implied volatility of the underlying future.

However, Bolviken and Benth (2000), argue that the family of Normal Inverse Gaussian (NIG) distribution is able to portray stochastic phenomena that have heavy tails or are strongly skewed. In addition to that, NIG distributions are not confined to the positive half axis. Therefore, with the NIG distribution the financial analyst has at its disposal a model that can be adapted to many different shapes while the distribution of sums of independent random variables are still trivial to compute.

III. METHODOLOGY

3.1 .1 Levy Process

Definition 1.1 A cadlag real valued stochastic process $\{X(t)\}_{t\geq 0}$ such that $X(0)=0$ is called a Levy Process if it has stationary independent increments and is stochastically continuous.

Brownian motion

The *Brownian motion* is a Levy process that has a drift. The standard Browniam motion follows a normal distribution with mean μ and variance σ^2 .

3.1.2 Normal Inverse Gaussian process (NIG)

The *Normal inverse Gaussian process* (NIG) is a Levy process $\{X(t)\}_{t\geq 0}$ that has normal inverse Gaussian distributed increments. Specifically, $X(t)$ has a $NIG(\alpha,\beta,\delta t,\mu t)-$ distribution with parameters $\alpha>0, |\beta|<\alpha, \delta>0$ and $\mu\in\mathbb{R}$.

The $NIG(\alpha,\beta,\delta,\mu)-$ distribution has a probability density function

$$f_{NIG(x;\alpha,\beta,\delta,\mu)}=\frac{\alpha\delta}{\pi}\frac{K_1(\alpha\sqrt{\delta^2-(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}\exp\{\delta\sqrt{\alpha^2-\beta^2+\beta(x-\mu)}\}$$

Where

$$K_n(Z)=\frac{1}{2}\int_0^\infty u^{\upsilon-1}\exp\{-\frac{Z}{2}(u+\frac{1}{u})\}du$$

is the modified Bessel function of the third kind, while the characteristic function is given by

$$\phi_{NIG}(u)=\exp(-\delta(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2}))e^{i\mu u}.$$

The mean, variance, skewness and kurtosis of NIG distribution are

$$\text{Mean}=\mu+\frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}$$

$$\text{Variance}=\frac{\delta\alpha^2}{(\sqrt{\alpha^2-\beta^2})^3}$$

$$\text{Skewness}=\frac{3\beta}{\alpha(\delta\sqrt{\alpha^2-\beta^2})^{\frac{1}{2}}}$$

$$\text{Kurtosis}=3\left(1+\frac{\alpha^2+4\beta^2}{\delta\alpha^2\sqrt{\alpha^2-\beta^2}}\right)$$

3.1.3 Symmetric Normal Inverse Gaussian distribution

The Symmetric NIG Levy process has symmetric NIG marginals. The NIG distribution is symmetric when the skewness parameter $\beta = 0$. In this case, the density of a symmetric NIG is

$$f_{NIG}(x)=\frac{\alpha}{\pi}e^{\alpha\delta}\frac{K_1\left(\alpha\delta\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1+\left(\frac{x-\mu}{\delta}\right)^2}}$$

It follows from the equations of the mean, variance and kurtosis, that μ is mean, $\frac{\delta}{\alpha}$ is variance and $3+\frac{3}{\alpha\delta}$ is kurtosis. We will denote the distribution of symmetric NIG by $SNIG(\alpha,0,\delta,\mu)$

The characteristic function for symmetric NIG is

$$\phi(u)=e^{iu\mu}e^{\alpha\delta\left(1-\sqrt{1+\left(\frac{u}{\alpha}\right)^2}\right)} \qquad \text{(i)}$$

By an inspection of the characteristic function we can see that the characteristic generator of symmetric Normal inverse Gaussian distribution is given by

$$\psi(\upsilon)=e^{\varsigma\left(1-\sqrt{1+\frac{2\upsilon}{\varsigma}}\right)} \qquad \text{(ii)}$$

where, $\varsigma=\alpha\delta$

3.2 BASIC MODELLING OF THE COMPLIANCE EVENT (see Carmona and Hinz(2011))

In the one-period setting, credits are allocated at the beginning of the period to enable allowance trading until time T and to encourage agents to exercise efficient abatement strategies. At the compliance date T, market participants cover their emissions by redeeming allowances or pay a penalty π per unit of pollution not offset by credits. In this one-period model, unused allowances expire and are worthless because we do not allow for banking into the next period. At compliance date T the allowance price S_T is a random variable taking only the values 0 and π. More precisely, if the market remains under the target pollution level, then the price approaches zero. Otherwise, the allowance price tends to the penalty level π.

All the relevant asset price evolutions are assumed to be given by adapted stochastic processes on a filtered probability space $\left(\Omega, F, \mathrm{P}, (F_t)_{t\in[0,T]}\right)$ on which we fix an equivalent probability measure Q ~P which we call the spot martingale measure.

We denote by $(S_t)_{t\in[0,T]}$ the price process of a future contract with maturity date T written on the allowance price. Given the digital nature of the terminal allowance price, S_T the central object of our study is the event $N \subset F_T$ of non-compliance which settles the $\{0,\pi\}$-dichotomy of the terminal future prices by $S_T = \pi 1_N$. Furthermore, a standard no-arbitrage argument shows that the future prices $(S_t)_{t\in[0,T]}$ needs to be a martingale for the spot martingale measure ,Q. Hence, the problem of allowance price modeling reduces to the appropriate choice of the martingale

$$A_t = \pi \mathrm{E}^Q(1_N \mid F_t), t \in [0,T]$$

We choose our starting point to be the non-compliance event $N \in F_T$ which we describe as the event where a hypothetic positive-valued random variable Γ_T exceeds the boundary 1, say $N = \{\Gamma_T \geq 1\}$. If one denotes by E_T the total pollution within the period $[0,T]$ which must be balanced against the total number $\gamma \in (0,\infty)$ of credits issued by the regulator, then the event of non-compliance should be given by $N = \{E_T \geq \gamma\}$ which suggests that Γ_T should be viewed as the normalized total emission E_T / γ. However in our modeling, we merely describe the non-compliance event. Strictly speaking, so any random variable Γ_T with

$$\{\Gamma_T \geq 1\} = \{E_T / \gamma \geq 1\},$$

would do as well. On this account, we do not claim that Γ_T represents the total normalized emission E_T / γ. So the allowance Spot price is given by the martingale

$$A_t = \pi E^Q(1_{\{\Gamma_T \geq 1\}} \mid F_t), t \in [0,T] \qquad (1)$$

.where $A_t = S_t$ in this case. simplifying the notation, we consider the normalized futures price process

$$a_t := \frac{S_t}{\pi} = E^{\square}(1_{\{\Gamma_T \geq 1\}} \mid F_t), t \in [0,T]$$

The random variable Γ_T is chosen from a suitable parameterized family of random variables.

The random variable Γ_T modeled by geometric Brownian motion (gbm) is given by

$$\Gamma_T = \Gamma_0 e^{\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2 ds},$$

where W_t is a standard Brownian motion and σ is the volatility parameter. Since $\mu_{gbm} = 0$, Γ_T is a martingale with respect to the underlying Brownian motion.
which is given by,

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T \sigma_s^2 ds} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \qquad (2)$$

and it solves the stochastic differential equation

$$dS_t = \pi \Phi^{'}(\Phi^{-1}(S_t))\sqrt{Z_t}\, dW_t \qquad (3)$$

where the positive valued function $(0,T) \ni t \to Z_t$ is given by

$$Z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}, \; t \in [0,T] \qquad (4)$$

The martingale is a random variable taking only the values 0 and π and satisfies

$$P\{\lim_{t \to T} a_t \in \{0,1\}\} = 1 \quad (6)$$

As an alternative model for Γ_T we introduce the geometric Levy process

$$\Gamma_T = \Gamma_0 \exp(\mu_{nig}(t) + Y_t)$$

where Y_t is a normal inverse Gaussian Levy process and $Y_T = \sum_{t=1}^{T} \Delta Y_t$, where $t = 1,\ldots,N$ are i.i.d.

If we specify the parameters at time t=1, the first two moments (expectation and variance) are

$$E[Y_1] = \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}, \qquad Var[Y_1] = \frac{\delta\alpha^2}{\left(\alpha^2 - \beta^2\right)^{3/2}}$$

This means that the log returns with time increment equal to 1 have expectation $\mu_{nig} + E[Y_1]$ (or volatility) $Var[Y_1]$ when they are modeled as independent normally inverse Gaussian distributed variables. We should note that when using the NIG process for option pricing, the location parameter of the distribution has no effect on the option value, so for convenience we will take $\mu_{nig} = 0$

Since we have an infinitely divisible characteristic function, we can define the NIG process $Y_1 = \{Y_t, t \geq 0\}$, which starts at zero and has independent and stationary increments each with a $NIG(\alpha, \beta, \delta)$ distribution and the entire process has a $NIG(\alpha, \beta, \delta t)$ law.

Note: $Y_t = Y_{(T(t))}$ for each $t \geq 0$, where $T(t)$ is an inverse Gaussian subordinator, $IG(l; a, b)$, which is independent of the standard Brownian motion with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$

We have the following connection between the parameters in the geometric Brownian motion and normal inverse Gaussian models

$$\mu_{gbm} = \mu_{nig} + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}, \qquad \sigma^2 = \frac{\delta\alpha^2}{\left(\alpha^2 - \beta^2\right)^{3/2}}.$$

The Levy-Kintchine representation of Y_t is

$$Y_t = \xi t + \int_0^t \int_{R\setminus\{0\}} z \tilde{N}(dt, dz), \; \xi = E[Y_1],$$

where $N(dt,dz)$ is a Poisson random measure and $\tilde{N}(dt,dz) = N(dt,dz) - dt \times n(dz)$ is the compensated Poisson random measure associated to Y_t. The levy measure of Y_t is

$$n(dz) = \frac{\delta\alpha}{\pi|z|}\exp(\beta z)K_1(\alpha|z|)dz.$$

It follows from the Levy-Khintchine representation that the normal inverse Gaussian Levy process is a pure jump process.

In the symmetric case (i.e., when $\beta = 0$), we have $\mu_{gbm} = \mu_{nig}$ and $\sigma^2 = \frac{\delta}{\alpha}$. Y_t is a martingale with respect to its own filtration when $E[Y_1] = \xi = 0,$ which is equivalent to $\beta = 0$.

Therefore

$$\Gamma_t = \Gamma_0 \exp(Y_t) \quad (5)$$

$$\begin{aligned} a_t &= E^Q\left(1_{\{\Gamma_T \geq 1\}} \mid F_t\right) = Q\{\Gamma_T \geq 1 \mid F_t\} \\ &= Q\left\{\Gamma_0 e^{Y_t} \geq 1 \mid F_t\right\} \end{aligned} \quad (6)$$

The characteristic triplet of Y_t is $(\mu, c, n(d(z)))$, where c is used to define the Brownian component. In this case we will assume that $c \neq 0$.

We denote by $S(\mu,\sigma^2,\psi)$ the distribution of Y_1 whose characteristic function is of the form

$$\varphi_{Y_1}(u) = e^{iu\mu}\psi\left(\frac{\sigma^2}{2}u^2\right), \quad (7)$$

The function $\psi(u):[0,\infty]$ is called the characteristic generator. It is unique up to scaling and if chosen such that $\psi'(0) = -1,$ yields that μ and σ^2 are the mean and variance of Y_1 respectively.

UnderQ , Y_t remains a symmetric NIG Levy process with characteristic triplet $\left(\tilde{\mu}, \sigma^2, \psi\right)$, where

$$\tilde{\mu}_1 = r - \ln\left(-\sigma^2/2\right)$$

r is the risk free rate.

Now it is easy to see that Q_1 is a natural equivalent martingale measure. Indeed, since $-Y_t$ is a Levy process with P-characteristic triplets $(-\mu, c, n(dz))$ and since the distribution of $-Y_1$ is $S(-\mu,\sigma^2,\psi)$, Q_1 is chosen so that

$$\tilde{\mu}_1 = r + \ln\left(-\sigma^2/2\right)$$

By the uniqueness of $(\tilde{\mu}_1)$, Q_1 is unique.

Under Q_1, Y_t is a symmetric NIG Levy process with marginal distribution from the family $S(\tilde{\mu}t, \sigma t, \psi_t)$

Proposition: Denote by F_T the P-distribution function of the standardized variable $(Y_T - \mu T)/(\sigma\sqrt{T})$:

$$F_T(y) = P\left(Y_T \le \sigma\sqrt{T}y + \mu T\right).$$

Then

$$F_T(y) = Q\left(Y_T \le \sigma\sqrt{T}y + \tilde{\mu}T\right) = Q_1\left(Y_T \le \sigma\sqrt{T}y + \tilde{\mu}_1 T\right)$$

We will approximate the standardized symmetric NIG distribution by the standard normal.

$$\begin{aligned} a_t &= E^Q\left(1_{\{\Gamma_T \ge 1\}} \mid F_t\right) = Q\{\Gamma_T \ge 1 \mid F_t\} \\ &= \Phi\left(\frac{\ln\Gamma_0 + (r + \ln\psi(-\sigma^2/2))t}{\sigma\sqrt{t}}\right) \end{aligned} \qquad (8)$$

By (ii) and $\zeta = \alpha\delta = \alpha^2\sigma^2$, we obtain

$$\ln\psi\left(-\frac{\sigma^2}{2}\right) = \zeta\left(1 - \sqrt{1 - \frac{\sigma^2}{\zeta}}\right) = \alpha^2\sigma^2 - \alpha\sigma^2\sqrt{\alpha^2 - 1}.$$

Equation (8) becomes

$$a_t = \Phi\left(\frac{\ln\Gamma_0 + (r + \alpha^2\sigma^2 - \alpha\sigma^2\sqrt{\alpha^2 - 1})t}{\sigma\sqrt{t}}\right), \qquad (9)$$

Remark: If the excess emissions have a normal distribution (Normal inverse Gaussian) then the price process automatically becomes a normal distribution (Normal inverse Gaussian).

3.3 Estimation of Model Parameters

Parameter estimates that are such that the model lead to best fit to the carbon price data are determined . By using the NIG model, it means that we have four parameters that need to be estimated. While for the normal distribution, only two parameters need to be estimated. Given data we obtain these estimates for the parameters.

To obtain estimates of the parameters , maximum likelihood estimation method is applied. Thus the likelihood function is maximized with respect to the parameters. Hence, we seek the parameter values that maximizes the likelihood function

$$L(\theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

We obtain the MLE estimate by maximizing the log likelihood function.The log likelihood function given a random sample of size n is given by

$$L = -n\ln(\pi) + n\ln(\alpha) + n(\partial\delta - \beta\mu) - \frac{1}{2}\sum_{i=1}^{n}\phi(x_i) + \beta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}K_1(\partial\alpha\phi(x_i)^{1/2})$$

The log likelihood function of the normal distribution, $N(\mu,\sigma)$ given a sample of size n is given by

$$L = \sigma^{-n}(2\pi)^{-n/2}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

The maximization of the log-likelihood function is done numerically using an optimization algorithm. For further details about this, see (Myung, 2003). However, I have chosen to use the R-package.

3.4 Goodness of fit

There are various approaches for measuring the goodness-of-fit of a given model. These include the following.

3.4.1 QQ-plots

We use the QQ-plot graphical technique for determining which distribution best fits the data set.

3.4.2 Anderson-Darling test statistic:

$$AD = \max_{x\in R}\frac{|F_n(x) - F(x)|}{\sqrt{F(x)(1-F(x))}}$$

Where, $F_n(x) =$ (no. of $x_i \le x$)/n, is the empirical cumulative distribution function and $F(x)$ is the cumulative distribution function.

A smaller value of AD means that the empirical distribution and fitted distribution are closer.

3.5 Risk-neutral Option Pricing

We assume that the price $B(t)$ of a risk-free asset satisfies the ordinary differential equation

$dB(t) = rB(t)dt$, where $r \ge 0$ is the interest rate.

By first *fundamental theorem of asset pricing*, if a *risk-neutral probability measure* exists, then there is no arbitrage. This risk-neutral probability is a martingale measure □ that is equivalent to the original probability measure P and such that the underlying asset price is a □ local martingale.

A *European call option* is the right but not obligation to buy a contingent claim at the *time of maturity* T to a fix *strike price* K. Thus the payoff function is given by

$$\max(S(T)-K,0).$$

The *arbitrage-free value* of the option at time $t < T$ can be defined as

$$C_t = e^{-r(T-t)}E^{□}\left[\max(S(T)-K,0)\right]$$

3.6 Option Pricing Using Fast Fourier Transform (FFT)

The European call option price will be based on the asset price process, S_t, with maturity time T and strike price K. Write $k = \log(K)$ and $s(T) = \log(S(T))$. $C_T(k)$ denote the option price and f_T the risk –neutral probability density function of price s_T.

The characteristic function of the density f_T is given by

$$\phi_T(u) = \int_{-\infty}^{\infty} e^{ius} f_T(s)ds. \quad (11)$$

The option value which is related to the risk-neutral density f_T is given by

$$C_T(k) = \int_{k}^{\infty} e^{-rT}(e^s - e^k) f_T(s)ds \quad (12)$$

Here $C_T(k)$ is not square integrable because when $k \to -\infty$ so that $K \to 0$, we have $C_T \to S(0)$. To obtain a square integrable function, we consider the modified price $c_T(k)$ given by

$$c_T(k) = e^{\lambda k} C_T(k), \quad (13)$$

for a suitable $\lambda > 0$. The value λ affects the speed of convergence.

The Fourier transform of $c_T(k)$ is defined by

$$\varphi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k)dk. \quad (14)$$

We first develop an analytical expression for $\varphi_T(v)$ in terms of characteristic function, ϕ_T, and then obtain call prices numerically using the inverse transform

$$\begin{aligned} C_T(k) &= \frac{\exp(-\lambda k)}{2\Pi} \int_{-\infty}^{\infty} e^{-ivk} \varphi_T(v)dv \\ &= \frac{\exp(-\lambda k)}{\Pi} \int_{0}^{\infty} e^{-ivk} \varphi_T(v)dv \end{aligned} \quad (15)$$

$$\begin{aligned} \varphi_T(v) &= \int_{-\infty}^{\infty} e^{ivk} \int_{k}^{\infty} e^{\lambda k} e^{-rT} \left(e^s - e^k\right) f_T(s) \\ &= \int_{-\infty}^{\infty} e^{-rT} f_T(s) \int_{-\infty}^{s} \left(e^{s+\lambda k} - e^{(1+\lambda)k}\right) e^{ivk} dkds \\ &= \int_{-\infty}^{\infty} e^{-rT} f_T(s) \left(\frac{e^{(\lambda+1+iv)s}}{\lambda + iv} - \frac{e^{(\lambda+1+iv)s}}{\lambda + 1 + iv} \right) ds \\ &= \frac{e^{-rT} \phi_T(v - (\lambda+1)i)}{\lambda^2 + \lambda - v^2 + i(2\lambda+1)v} \end{aligned} \quad (16)$$

Call values are determined by substituting (16) into (15) and performing the required integration. Since the FFT evaluates the integrand at $v = 0$, the use of $\exp(\lambda k)$ is required. Therefore it is necessary to make an appropriate choice of the coefficient λ. Positive values of λ assist in the integrability of the modified call value over the negative log strike axis, but aggravate the same condition for the positive log strike direction, and hence for it to be square-integrable as well, a sufficient condition is provided by $\varphi(0)$ being finite provided that $\phi_T\left(-(\lambda+1)i\right)$ is finite. From the definition of the characteristic function, this requires that

$$E[S_T^{\lambda+1}]<\infty \quad (17)$$

Carr and Madan suggest that, one may determine an upper bound on λ from the analytical expression for the characteristic function and the condition (17). One quarter of this upper bound serves as a good choice for λ, that is $\lambda \approx 0.75$.

At $v=0$ equation (15) becomes,

$$C_T(k)=\frac{\exp(-\lambda k)}{\Pi}\varphi_T(0)$$

Where, $\varphi_T(0)=\dfrac{e^{-rT}\phi_T(-(\lambda+1)i)}{\lambda^2+\lambda}$

Therefore, $C_T(k)=\dfrac{\exp(-\lambda k)}{\Pi}*\dfrac{e^{-rT}\phi_T(-(\lambda+1)i)}{\lambda^2+\lambda}$

The characteristic function of the log of S_T, which follows a NIG distribution, is given by

$$\phi_T(u)=\exp\left[\ln\pi\Phi(g)+T\left[r+\alpha^2\sigma^2-\alpha\sigma^2\sqrt{\alpha^2-iu}\right]\right]$$

$$\phi_T(-(\lambda+1)i)=\exp\left[\ln\pi\Phi(g)+T\left[r+\alpha^2\sigma^2-\alpha\sigma^2\sqrt{\alpha^2-(\lambda+1)}\right]\right]$$

The equation to get the option price of an NIG model is therefore,

$$C_T(k)=\frac{\exp(-\lambda k)}{\Pi}*\frac{\exp\left[\ln\pi\Phi(g)+T\left[r+\alpha^2\sigma^2-\alpha\sigma^2\sqrt{\alpha^2-(\lambda+1)}\right]\right]}{\left(\lambda^2+\lambda\right)}$$

We now get the expression for the characteristic function of the normal distribution and use it to get the option price

The characteristic function of normal distribution is given by

$$\phi_T(u)=e^{i\mu u-\frac{1}{2}\sigma^2u^2}$$

The Brownian motion is a type of Levy process. Therefore, under □ , S_t^N is a Levy process with characteristic triplet $\left(\tilde{\mu},c,\upsilon\right)$ and the distribution of S_1^N becomes $s\left(\tilde{\mu},\sigma,\psi\right)$, where

$$\tilde{\mu}=r-\ln\psi\left(-\frac{\sigma^2}{2}\right)$$

The standard Brownian motion is $N\left(0, \int_0^t \sigma^2 ds\right)$

Therefore, $\phi_T(u) = e^{-\frac{1}{2}\int_0^T \sigma_t^2 u^2 dt}$

$$\phi_T(-(\lambda+1)i) = \exp\left[\ln \pi\Phi(h) + T\left[r - \frac{1}{2}\int_0^T \sigma_t^2 (\lambda+1)^2 dt\right]\right]$$

The option valuation for the normal distribution is given by

$$C_T(k) = \frac{\exp(-\lambda k)}{\Pi} * \frac{\exp\left[\ln \pi\Phi(h) + T\left[r - \frac{1}{2}\int_0^T \sigma_t^2 (\lambda+1)^2 dt\right]\right]}{\left(\lambda^2 + \lambda\right)}$$

Remark: We will use AM92 Actuarial tables to get the values of $\Phi(g)$ and $\Phi(h)$.

IV. Application, Results and Conclusions

Data for analysis was sourced from [11], which is in the public domain. Codes for the main analysis appear in the Appendix. The results from the analysis are as follows.

4.1 Stylized features

4.1.1 Skewness and kurtosis

The skewness of log returns of price of carbon is -0.1212164. Since this value is near zero it is safe to say that our data is indeed symmetric. The value of kurtosis is -1.36154. This justifies the use of NIG levy process to model our data. The skewness and kurtosis of Gaussian distributions are 0 and 3 respectively.

4.2 Parameter Estimation

Asymmetric Normal Inverse Gaussian distribution:

Table 1: Maximum Likelihood Estimates for Parameters:

	μ	σ	β	α
NIG	1.562073917	0.003563	-1.650997	12.358663
Normal	-0.0952422	0.44958767		

4.3 Q-Q Plot

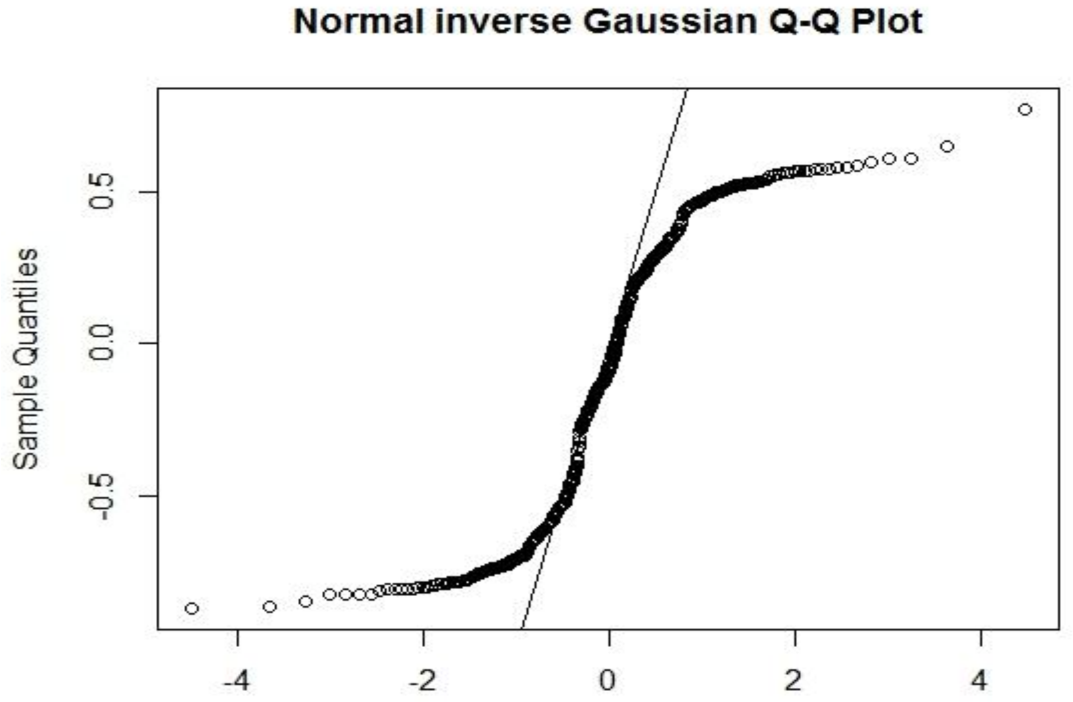

Figure 1.3: NIG QQ plot
param = (0, 1, 1, 0, -0.5)

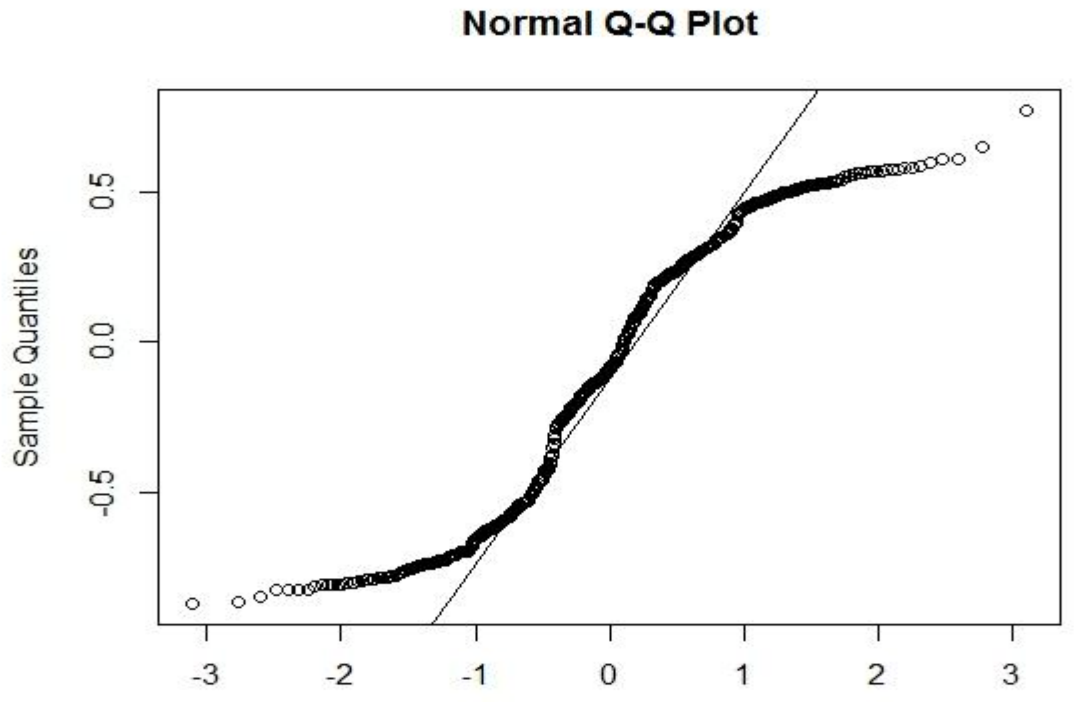

Figure 1.4: Normal QQ plot

The QQ-plots indicate that the empirical data fits much better to the NIG levy process than the Brownian motion.

4.4 Anderson-Darling (AD) test Statistic:

Table 2: AD statistic for NIG and Brownian Motion

	AD
NIG	131.7
Normal	208.5

The AD-statistic value is relatively smaller for the NIG Levy process than Brownian motion.

4.5 Model Parameter estimates

Suppose the strike price, K, is 20 Euros, 25 Euros or 30 Euros, T=4 (compliance phase), $\Gamma_0 = 5$, $\Pi = 3.14$ and r=0.05. The value of the call option using NIG model and Brownian motion are:

Table 3 : NIG model Parameter estimates

K	k	$C_T(k)$
20	2.9957322	1.2535
25	3.2188758	1.06
30	3.4011973	0.92

Table 4 : Brownian Motion Model Parameter estimates

K	k	$C_T(k)$
20	2.9957322	0.00806
25	3.2188758	0.00682
30	3.4011973	0.00595

The option prices in the NIG model are higher than those of the Brownian model with respect to the given strike prices. As the strike price increase, the value of the option decreases in both models.

V. Conclusion

The significant parameters in the NIG model are the ∝ and σ, while for the Brownian Motion model the significant parameter is σ . The NIG model has a better fit to data compared to the Brownian Motion model. Using the Fast Fourier transform, NIG model gives higher option prices than the Brownian motion.

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[11.] Data Link: http://www.investing.com/commodities/carbon-emissions-historical-data

Appendix: R codes used in the Analysis

```
Carbon = read.csv(file.choose())

Carbon

Price <-carbon$Price

Price

Plot(price, xlab="Fig1.1:carbon prices")

Plot (log_returns4, type="1", xlab="Figure 2.2: Log returns of carbon prices", ylab = "Log returns ")

Log_returns4 < -diff(log(price), lag=364 )

Log_returns4

par(mfrow=c(1,2))

qqnig(log_returns4, mu=0, delta=1, alpha=1, beta=0, xlab="Figure 1.3: NIG QQ plot")

qqnorm(log_returns4, xlab = "  Figure 1.4 : Normal QQ plot ")

qqline (log_returns4)

y1

ad.test(y1)
```